

Embedded Gaussian Unitary Ensembles with $U(\Omega) \otimes SU(r)$ Embedding generated by Random Two-body Interactions with $SU(r)$ Symmetry

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Abstract

Following the earlier studies on embedded unitary ensembles generated by random two-body interactions [EGUE(2)] with spin $SU(2)$ and spin-isospin $SU(4)$ symmetries, developed is a general formulation, for deriving lower order moments of the one- and two-point correlation functions in eigenvalues, that is valid for any EGUE(2) and BEGUE(2) ('B' stands for bosons) with $U(\Omega) \otimes SU(r)$ embedding and with two-body interactions preserving $SU(r)$ symmetry. Using this formulation with $r = 1$, we recover the results derived by Asaga et al [Ann. Phys. (N.Y.) **297**, 344 (2002)] for spinless boson systems. Going further, new results are obtained for $r = 2$ (this corresponds to two species boson systems) and $r = 3$ (this corresponds to spin 1 boson systems).

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I. INTRODUCTION

A long standing question for the embedded ensembles is about their analytical tractability. Amenability to mathematical treatment is one of the four conditions laid down by Dyson [1] for the validity of a random matrix ensemble. Simplest of the two-body unitary ensemble is the embedded Gaussian unitary ensemble of two-body interactions [EGUE(2)] for spinless fermion systems. For m fermions in N sp states, the embedding is generated by the $SU(N)$ algebra and although this ensemble is known for many years, only recently [2], after the first indications implicit in [3, 4], it is established that the $SU(N)$ Wigner-Racah algebra solves EGUE(2) and also the more general EGUE(k) [as well as EGOE(k)]. These results, with $U(N)$ algebra, extended to BEGUE(k) for spinless bosons in N sp states (see [2, 5]). For EGUE(2)-s for fermions with spin and EGUE(2)- $SU(4)$ for fermions with Wigner's spin-isospin $SU(4)$ symmetry, the embedding algebras, with Ω number of spatial degrees of freedom for a single fermion, are $U(\Omega) \otimes SU(2)$ and $U(\Omega) \otimes SU(4)$ respectively. It was shown in [6, 7] that the Wigner-Racah algebra of these embedding algebras will allow one to obtain analytical results for the lower order moments of the one- and two-point correlation functions in eigenvalues. Similarly, following the recent work [8, 9] on BEGOEs, it is easy to recognize that the embedding algebras for BEGUE(2)- F for two-species boson systems with F -spin and BEGUE(2)- $SU(3)$ for spin one boson systems are $U(\Omega) \otimes SU(2)$ and $U(\Omega) \otimes SU(3)$ respectively. The purpose of the present paper is to establish on one hand that the Wigner-Racah algebra of these embedding algebras solve the corresponding embedded unitary ensembles and on the other to generalize the formalism to any EGUE(2) with $U(\Omega) \otimes SU(r)$ embedding and generated by random two-body interaction with $SU(r)$ symmetry. Hereafter we call these ensembles EGUE(2)- $SU(r)$ and they apply to both fermion and boson systems.

In Section 2, given is the general formulation based on Wigner-Racah algebra for lower order moments of the one- and two-point functions in eigenvalues generated by EGUE(2)- $SU(r)$ (r is any positive integer, $r \geq 1$). Sections 3, 4 and 5 give analytical results for boson systems with $r = 1$, $r = 2$ and $r = 3$ respectively. In addition, some numerical results for lower order correlations generated by these ensembles are also given in Section 5. Finally, Section 6 gives concluding remarks.

II. EGUE(2)- $SU(r)$ ENSEMBLES: GENERAL FORMULATION

Consider a system of m fermions or bosons in Ω number of sp levels each r -fold degenerate. Then the SGA is $U(r\Omega)$ and it is possible to consider $U(r\Omega) \supset U(\Omega) \otimes SU(r)$ algebra. Now, for random two-body Hamiltonians preserving $SU(r)$ symmetry, one can introduce embedded GUE with $U(\Omega) \otimes SU(r)$ embedding and this ensemble is called EGUE(2)- $SU(r)$. Ensembles with $r = 2, 4$ for fermions correspond to fermions with spin and spin-isospin $SU(4)$ symmetry. Similarly, for bosons $r = 2, 3$ are of interest. Also $r = 1$ gives back EGUE(2) and BEGUE(2) both. It is important to note that the distinction between fermions and bosons is in the $U(\Omega)$ irreps that need to be considered. Now we will give a formulation in terms of $SU(\Omega)$ Wigner-Racah algebra (the $SU(r)$ algebra involved will be simple as H has $SU(r)$ symmetry) that is valid for any r . The discussion in the remaining part of this Section is essentially from [7] but it is repeated briefly not only for completeness but also to generalize it to any r and also to bosons systems (in [7], fermions with $r = 4$ is used).

Let us begin with normalized two-particle states $|f_2 F_2; v_2 \beta_2\rangle$ where the $U(r)$ irreps $F_2 = \{1^2\}$ and $\{2\}$ and the corresponding $U(\Omega)$ irreps f_2 are $\{2\}$ (symmetric) and $\{1^2\}$ (antisymmetric) respectively for fermions and $\{1^2\}$ (antisymmetric) and $\{2\}$ (symmetric) respectively for bosons. Similarly v_2 are additional quantum numbers that belong to f_2 and β_2 belong to F_2 . As f_2 uniquely defines F_2 , from now on we will drop F_2 unless it is explicitly needed and also we will use the $f_2 \leftrightarrow F_2$ equivalence whenever needed. With $A^\dagger(f_2 v_2 \beta_2)$ and $A(f_2 v_2 \beta_2)$ denoting creation and annihilation operators for the normalized two particle states, a general two-body Hamiltonian operator \hat{H} preserving $SU(r)$ symmetry can be written as

$$\hat{H} = \hat{H}_{\{2\}} + \hat{H}_{\{1^2\}} = \sum_{f_2, v_2^i, v_2^f, \beta_2; f_2 = \{2\}, \{1^2\}} H_{f_2 v_2^i v_2^f}(2) A^\dagger(f_2 v_2^f \beta_2) A(f_2 v_2^i \beta_2). \quad (1)$$

In Eq. (1), $H_{f_2 v_2^i v_2^f}(2) = \langle f_2 v_2^f \beta_2 | H | f_2 v_2^i \beta_2 \rangle$ independent of the β_2 's. The uniform summation over β_2 in Eq. (1) ensures that \hat{H} is $SU(r)$ scalar and therefore it will not connect states with different f_2 's. However, \hat{H} is not a $SU(r)$ invariant operator. Just as the two particle states, we can denote the m particle states by $|f_m v_m^f \beta_m^F\rangle$; $F_m = \tilde{f}_m$ for fermions and $F_m = f_m$ for bosons. Action of \hat{H} on these states generates states that are degenerate with respect to β_m^F but not v_m^f . Therefore for a given f_m , there will be $d_\Omega(f_m)$ number of levels

each with $d_r(\tilde{f}_m)$ number of degenerate states. Formula for the dimension $d_\Omega(f_m)$ is [10],

$$d_\Omega(f_m) = \prod_{i < j=1}^{\Omega} \frac{f_i - f_j + j - i}{j - i}, \quad (2)$$

where, $f_m = \{f_1, f_2, \dots\}$. Equation (2) also gives $d_r(F_m)$ with the product ranging from $i = 1$ to r and replacing f_i by F_i . As \hat{H} is a $SU(r)$ scalar, the m particle H matrix will be a direct sum of matrices with each of them labeled by the f_m 's with dimension $d_\Omega(f_m)$. Thus

$$H(m) = \sum_{f_m} H_{f_m}(m) \oplus . \quad (3)$$

It should be noted that the matrix elements of $H_{f_m}(m)$ matrices receive contributions from both $H_{\{2\}}(2)$ and $H_{\{1^2\}}(2)$.

Embedded random matrix ensemble EGUE(2)- $SU(r)$ for a m fermion or boson system with a fixed f_m , i.e. $\{H_{f_m}(m)\}$, is generated by the ensemble of H operators given in Eq. (1) with $H_{\{2\}}(2)$ and $H_{\{1^2\}}(2)$ matrices replaced by independent GUE ensembles of random matrices,

$$\{H(2)\} = \{H_{\{2\}}(2)\}_{GUE} \oplus \{H_{\{1^2\}}(2)\}_{GUE} . \quad (4)$$

In Eq. (4), $\{--\}$ denotes ensemble. Random variables defining the real and imaginary parts of the matrix elements of $H_{f_2}(2)$ are independent Gaussian variables with zero center and variance given by (with bar representing ensemble average),

$$\overline{H_{f_2 v_2^1 v_2^2}(2) H_{f_2' v_2^3 v_2^4}(2)} = \delta_{f_2 f_2'} \delta_{v_2^1 v_2^4} \delta_{v_2^2 v_2^3} (\lambda_{f_2})^2 . \quad (5)$$

Also, the independence of the $\{H_{\{2\}}(2)\}$ and $\{H_{\{1^2\}}(2)\}$ GUE ensembles imply,

$$\overline{\left[H_{\{2\} v_2^1 v_2^2}(2) \right]^P} \overline{\left[H_{\{1^2\} v_2^3 v_2^4}(2) \right]^Q} = \left\{ \overline{\left[H_{\{2\} v_2^1 v_2^2}(2) \right]^P} \right\} \left\{ \overline{\left[H_{\{1^2\} v_2^3 v_2^4}(2) \right]^Q} \right\} \quad (6)$$

for P and Q even and zero otherwise. Action of \hat{H} defined by Eq. (1) on m particle basis states with a fixed f_m , along with Eqs. (5)-(6) generates EGUE(2)- $SU(r)$ ensemble $\{H_{f_m}(m)\}$; it is labeled by the $U(\Omega)$ irrep f_m with matrix dimension $d_\Omega(f_m)$.

As shown in [2, 6, 7], tensorial decomposition of \hat{H} with respect to the embedding algebra $U(\Omega) \otimes SU(r)$ plays a crucial role in generating analytical results; as before $U(\Omega)$ and $SU(\Omega)$ are used interchangeably. As \hat{H} preserves $SU(r)$, it transforms as the irrep $\{0\}$ with respect to the $SU(r)$ algebra. However with respect to $SU(\Omega)$, the tensorial characters, in Young

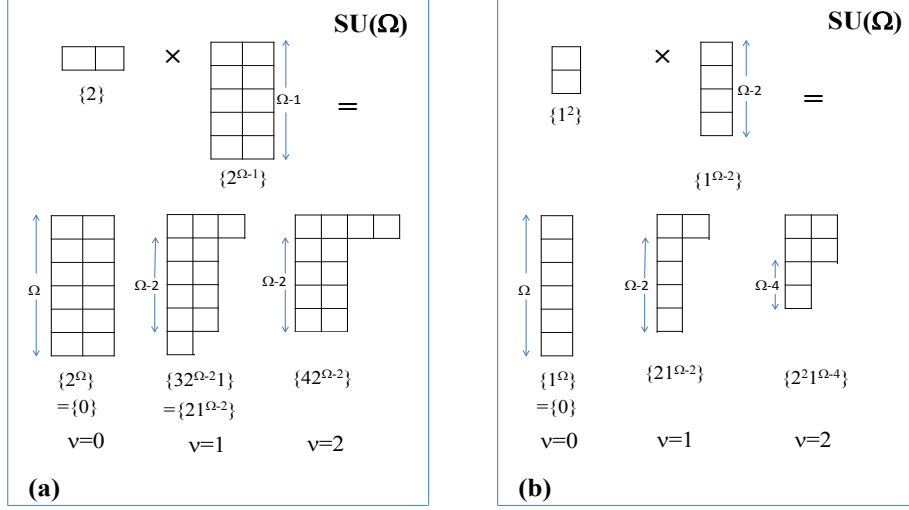


FIG. 1. Young tableaux for the tensorial parts of a two-body Hamiltonian with respect to $SU(\Omega)$ algebra. Young tableaux for various (a) tensorial parts with respect to $SU(\Omega)$ for the $f_2 = \{2\}$ part of H ; (b) tensorial parts with respect to $SU(\Omega)$ for the $f_2 = \{1^2\}$ part of H .

tableaux notation, for $f_2 = \{2\}$ are $\mathbf{F}_\nu = \{0\}$, $\{21^{\Omega-2}\}$ and $\{42^{\Omega-2}\}$ with $\nu = 0, 1$ and 2 respectively. Similarly for $f_2 = \{1^2\}$ they are $\mathbf{F}_\nu = \{0\}$, $\{21^{\Omega-2}\}$ and $\{2^21^{\Omega-4}\}$ with $\nu = 0, 1, 2$ respectively. Note that $\mathbf{F}_\nu = f_2 \times \overline{f_2}$ where $\overline{f_2}$ is the irrep conjugate to f_2 and the \times denotes Kronecker product. Given a $U(\Omega)$ irrep $\{f\} = \{f_1, f_2, \dots, f_\Omega\}$, we have $\overline{f} = \{f_1 - f_\Omega, f_1 - f_{\Omega-1}, \dots, f_1 - f_2, 0\}$. Young tableaux for the F_ν 's are shown in Fig. 1. Now, we can define unitary tensors B 's that are scalars in $SU(r)$ space,

$$B(f_2 \mathbf{F}_\nu \omega_\nu) = \sum_{v_2^i, v_2^f, \beta_2} A^\dagger(f_2 v_2^f \beta_2) A(f_2 v_2^i \beta_2) \left\langle f_2 v_2^f \overline{f_2} \overline{v_2^i} \mid \mathbf{F}_\nu \omega_\nu \right\rangle \times \left\langle F_2 \beta_2 \overline{F_2} \overline{\beta_2} \mid 00 \right\rangle. \quad (7)$$

In Eq. (7), $\langle f_2 --- \rangle$ are $SU(\Omega)$ Wigner coefficients and $\langle F_2 --- \rangle$ are $SU(r)$ Wigner coefficients. The expansion of \hat{H} in terms of B 's is,

$$\hat{H} = \sum_{f_2, \mathbf{F}_\nu, \omega_\nu} W(f_2 \mathbf{F}_\nu \omega_\nu) B(f_2 \mathbf{F}_\nu \omega_\nu). \quad (8)$$

The expansion coefficients W 's follow from the orthogonality of the tensors B 's with respect to the traces over fixed f_2 spaces. Then we have the most important relation needed for all the results given ahead,

$$\overline{W(f_2 \mathbf{F}_\nu \omega_\nu) W(f_2' \mathbf{F}_\nu' \omega_\nu')} = \delta_{f_2 f_2'} \delta_{\mathbf{F}_\nu \mathbf{F}_\nu'} \delta_{\omega_\nu \omega_\nu'} (\lambda_{f_2})^2 d_r(F_2). \quad (9)$$

This is derived starting with Eq. (8) and using Eqs. (4)-(7) along with the sum rules for Wigner coefficients appearing in Eq. (7).

Turning to m particle H matrix elements, first we denote the $U(\Omega)$ and $U(r)$ irreps by f_m and F_m respectively. Correlations generated by EGUE(2)- $SU(r)$ between states with (m, f_m) and $(m', f_{m'})$ follow from the covariance between the m -particle matrix elements of H . Now using Eqs. (8) and (9) along with the Wigner-Eckart theorem applied using $SU(\Omega) \otimes SU(r)$ Wigner-Racah algebra (see for example [11]) will give

$$\begin{aligned}
& \overline{H_{f_m v_m^i v_m^f} H_{f_{m'} v_{m'}^i v_{m'}^f}} \\
&= \overline{\langle f_m F_m v_m^f \beta \mid H \mid f_m F_m v_m^i \beta \rangle \langle f_{m'} F_{m'} v_{m'}^f \beta' \mid H \mid f_{m'} F_{m'} v_{m'}^i \beta' \rangle} \\
&= \sum_{f_2, \mathbf{F}_\nu, \omega_\nu} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} \sum_{\rho, \rho'} \langle f_m \mid \mid B(f_2 \mathbf{F}_\nu) \mid \mid f_m \rangle_\rho \langle f_{m'} \mid \mid B(f_2 \mathbf{F}_\nu) \mid \mid f_{m'} \rangle_{\rho'} \\
&\quad \times \langle f_m v_m^i \mathbf{F}_\nu \omega_\nu \mid f_m v_m^f \rangle_\rho \langle f_{m'} v_{m'}^i \mathbf{F}_\nu \omega_\nu \mid f_{m'} v_{m'}^f \rangle_{\rho'} ; \\
&\langle f_m \mid \mid B(f_2 \mathbf{F}_\nu) \mid \mid f_m \rangle_\rho = \sum_{f_{m-2}} F(m) \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} \frac{U(f_m \bar{f}_2 f_m f_2; f_{m-2} \mathbf{F}_\nu)_\rho}{U(f_m \bar{f}_2 f_m f_2; f_{m-2} \{0\})} .
\end{aligned} \tag{10}$$

Here the summation in the last equality is over the multiplicity index ρ and this arises as $f_m \times \mathbf{F}_\nu$ gives in general more than once the irrep f_m . In Eq. (10),

$$F(m) = -m(m-1)/2 , \tag{11}$$

$d_\Omega(f_m)$ is given by Eq. (2) and $\langle \dots \rangle$ and $U(\dots)$ are $SU(\Omega)$ Wigner and Racah coefficients respectively. Similarly, \mathcal{N}_{f_m} is dimension with respect to the S_m group [10],

$$\mathcal{N}_{f_m} = \frac{m! \prod_{i < k=1}^p (\ell_i - \ell_k)}{\ell_1! \ell_2! \dots \ell_p!} ; \quad \ell_i = f_i + p - i . \tag{12}$$

Note that p denotes total number of rows in the Young tableaux for f_m .

Lower order cross correlations between states with different (m, f_m) are given by the normalized bivariate moments $\Sigma_{PQ}(m, f_m : m', f_{m'})$, $P = Q = 1, 2$ of the two-point function

S^ρ where, with $\rho^{m,f_m}(E)$ defining fixed- (m, f_m) density of states,

$$S^{mf_m:m'f_{m'}}(E, E') = \overline{\rho^{m,f_m}(E)\rho^{m',f_{m'}}(E')} - \overline{\rho^{m,f_m}(E)} \overline{\rho^{m',f_{m'}}(E')} ;$$

$$\Sigma_{11}(m, f_m : m', f_{m'}) = \overline{\langle H \rangle^{m,f_m} \langle H \rangle^{m',f_{m'}}} / \sqrt{\overline{\langle H^2 \rangle^{m,f_m} \langle H^2 \rangle^{m',f_{m'}}}} , \quad (13)$$

$$\Sigma_{22}(m, f_m : m', f_{m'}) = \overline{\langle H^2 \rangle^{m,f_m} \langle H^2 \rangle^{m',f_{m'}}} / \left[\overline{\langle H^2 \rangle^{m,f_m} \langle H^2 \rangle^{m',f_{m'}}} \right] - 1 .$$

In Eq. (13), $\overline{\langle H^2 \rangle^{m,f_m}}$ is the second moment (or variance) of the eigen value density $\overline{\rho^{m,f_m}(E)}$ and its centroid $\overline{\langle H \rangle^{m,f_m}} = 0$ by definition. As $\langle H \rangle^{m,f_m}$ is the trace of H (divided by dimensionality) in (m, f_m) space, only $\mathbf{F}_\nu = \{0\}$ will generate $\overline{\langle H \rangle^{m,f_m} \langle H \rangle^{m',f_{m'}}}$. Then trivially,

$$\overline{\langle H \rangle^{m,f_m} \langle H \rangle^{m',f_{m'}}} = \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} P^{f_2}(m, f_m) P^{f_2}(m', f_{m'}) ; \quad (14)$$

$$P^{f_2}(m, f_m) = F(m) \sum_{f_{m-2}} \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} .$$

Writing $\overline{\langle H^2 \rangle^{m,f_m}}$ explicitly in terms of m particle H matrix elements,

$$\overline{\langle H^2 \rangle^{m,f_m}} = [d(f_m)]^{-1} \sum_{v_m^1, v_m^2} \overline{H_{f_m v_m^1 v_m^2} H_{f_m v_m^2 v_m^1}} ,$$

and applying Eq. (10) and the orthonormal properties of the $SU(\Omega)$ Wigner coefficients lead to

$$\overline{\langle H^2 \rangle^{m,f_m}} = \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} \sum_{\nu=0,1,2} \mathcal{Q}^\nu(f_2 : m, f_m) \quad (15)$$

where

$$\mathcal{Q}^\nu(f_2 : m, f_m) = [F(m)]^2 \sum_{f_{m-2}, f'_{m-2}} \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} \frac{\mathcal{N}_{f'_{m-2}}}{\mathcal{N}_{f_m}} X_{UU}(f_2; f_{m-2}, f'_{m-2}; \mathbf{F}_\nu) . \quad (16)$$

The X_{UU} function involves $SU(\Omega)$ Racah coefficients,

$$X_{UU}(f_2; f_{m-2}, f'_{m-2}; \mathbf{F}_\nu) = \sum_\rho \frac{U(f_m, \overline{f_2}, f_m, f_2; f_{m-2}, \mathbf{F}_\nu)_\rho U(f_m, \overline{f_2}, f_m, f_2; f'_{m-2}, \mathbf{F}_\nu)_\rho}{U(f_m, \overline{f_2}, f_m, f_2; f_{m-2}, \{0\}) U(f_m, \overline{f_2}, f_m, f_2; f'_{m-2}, \{0\})} . \quad (17)$$

Summation over the multiplicity index ρ in Eq. (17) arises naturally in applications to physical problems as all the physically relevant results should be independent of ρ which is

a label for equivalent $SU(\Omega)$ irreps. It is easy to see that,

$$\mathcal{Q}^{\nu=0}(f_2 : m, f_m) = [P^{f_2}(m, f_m)]^2 . \quad (18)$$

Eqs. (14)-(16) and Table 4 of [7] will allow us to calculate covariances Σ_{11} in energy centroids; Table 4 of [7] is a simplified version of the tables in [12]. For the covariances Σ_{22} in spectral variances, the formula is [7]

$$\begin{aligned} \Sigma_{22}(m, f_m; m', f_{m'}) &= \frac{X_{\{2\}} + X_{\{1^2\}} + 4X_{\{1^2\}\{2\}}}{\langle H^2 \rangle^{m, f_m} \langle H^2 \rangle^{m', f_{m'}}} ; \\ X_{f_2} &= \frac{2(\lambda_{f_2})^4}{[d_\Omega(f_2)]^2} \sum_{\nu=0,1,2} [d_\Omega(\mathbf{F}_\nu)]^{-1} \mathcal{Q}^\nu(f_2 : m, f_m) \mathcal{Q}^\nu(f_2 : m', f_{m'}) , \\ X_{\{1^2\}\{2\}} &= \frac{\lambda_{\{2\}}^2 \lambda_{\{1^2\}}^2}{d(\{2\})d(\{1^2\})} \sum_{\nu=0,1} [d_\Omega(\mathbf{F}_\nu)]^{-1} \mathcal{R}^\nu(m, f_m) \mathcal{R}^\nu(m', f_{m'}) . \end{aligned} \quad (19)$$

Here $d_\Omega(\mathbf{F}_\nu)$ is the dimension of the irrep \mathbf{F}_ν , and we have $d_\Omega(\{0\}) = 1$, $d_\Omega(\{2, 1^{\Omega-2}\}) = \Omega^2 - 1$, $d_\Omega(\{4, 2^{\Omega-2}\}) = \Omega^2(\Omega+3)(\Omega-1)/4$, and $d_\Omega(\{2^2, 1^{\Omega-4}\}) = \Omega^2(\Omega-3)(\Omega+1)/4$. Note that $\mathcal{Q}^\nu(f_2 : m, f_m)$ are defined in Eq. (16). The functions $\mathcal{R}^\nu(m, f_m)$ also involve $SU(\Omega)$ U -coefficients,

$$\begin{aligned} \mathcal{R}^\nu(m, f_m) &= [F(m)]^2 \sum_{f_{m-2}, f'_{m-2}} \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} \frac{\mathcal{N}_{f'_{m-2}}}{\mathcal{N}_{f_m}} Y_{UU}(f_{m-2}, f'_{m-2}; \mathbf{F}_\nu) ; \\ Y_{UU}(f_{m-2}, f'_{m-2}; \mathbf{F}_\nu) &= \sum_{\rho} \frac{U(f_m, \{1^{\Omega-2}\}, f_m, \{1^2\}; f_{m-2}, \mathbf{F}_\nu)_\rho U(f_m, \{2^{\Omega-1}\}, f_m, \{2\}; f'_{m-2}, \mathbf{F}_\nu)_\rho}{U(f_m, \{1^{\Omega-2}\}, f_m, \{1^2\}; f_{m-2}, \{0\}) U(f_m, \{2^{\Omega-1}\}, f_m, \{2\}; f'_{m-2}, \{0\})} . \end{aligned} \quad (20)$$

In $Y_{UU}(f_{m-2}, f'_{m-2}; \mathbf{F}_\nu)$, f_{m-2} comes from $f_m \otimes \{1^{\Omega-2}\}$ and f'_{m-2} comes from $f_m \otimes \{2^{\Omega-1}\}$. Similarly, the summation is over $\nu = 0$ and 1 only as $\nu = 2$ parts for $f_2 = \{2\}$ and $\{1^2\}$ are different. Formulas for Y_{UU} are given in Table 7 of [7] and they are simplified version of the results in [12]. It is useful to note that,

$$\mathcal{R}^{\nu=0}(m, f_m) = P^{\{2\}}(m, f_m) P^{\{1^2\}}(m, f_m) . \quad (21)$$

Compact analytical results collected in Tables 4 and 7 of [7] for X_{UU} and Y_{UU} and Eqs. (2), (12) - (21) will allow one to derive analytical/numerical results for spectral variances and covariances in energy centroids and variances for any EGUE(2)- $SU(r)$ for fermion or boson systems.

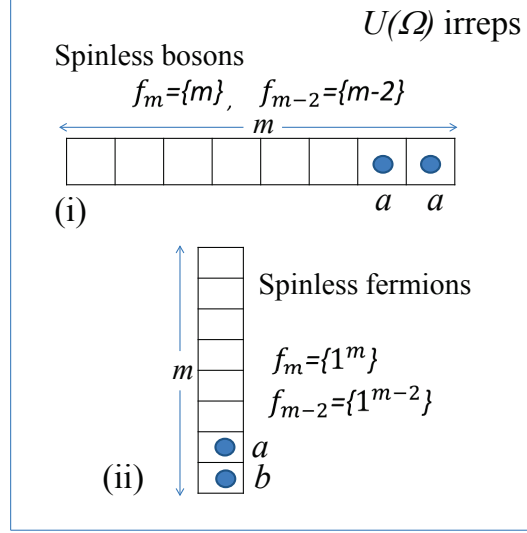


FIG. 2. Young tableaux denoting the $SU(\Omega)$ irreps $f_m = \{m\}$ and $\{1^m\}$ as appropriate for (i) spinless boson and (ii) spinless fermion systems. Removal of two boxes generating $m - 2$ particle irreps f_{m-2} for these systems are also shown in the figure. For (i) only the irrep $f_2 = \{2\}$ will apply and similarly for (ii) only $\{1^2\}$ will apply.

III. RESULTS FOR BEGUE(2): $r = 1$

Simplest of the EGUE(2)- $SU(r)$ are the EGUEs with $r = 1$ and they corresponds to EGUE(2) and BEGUE(2) depending on totally antisymmetric or symmetric f_m one considers. Also they correspond to $k = 2$ in [2] and [5] for fermion and boson systems respectively. As detailed results for fermion systems are available in [2, 6, 8], in the present Section and in the next two Sections we consider only boson systems. Let us begin with BEGUE(2). For this ensemble, in order to apply the formulas given Section 2 for $\langle H^2 \rangle$, Σ_{11} and Σ_{22} , first we need formulas for X_{UU} and Y_{UU} . Some of these, taken from Tables 4 and 7 of [7], are given in Table I by reducing them to much small number of formulas. For applying these formulas, we need the 'axial distances' τ_{ij} for the boxes i and j in a given Young tableaux.

Given a $f_m = \{f_1, f_2, \dots, f_\Omega\}$ we have,

$$\tau_{ij} = f_i - f_j + j - i . \quad (22)$$

In terms of τ_{ij} the functions $\Pi_a^{(b)}$, $\Pi_b^{(a)}$, $\Pi_a^{(bc)}$, Π'_a and Π''_a are defined as,

$$\begin{aligned} \Pi_a^{(b)} &= \prod_{i=1,2,\dots,\Omega; i \neq a, i \neq b} (1 - 1/\tau_{ai}) , \\ \Pi_b^{(a)} &= \prod_{i=1,2,\dots,\Omega; i \neq a, i \neq b} (1 - 1/\tau_{bi}) , \\ \Pi_a^{(bc)} &= \prod_{i=1,2,\dots,\Omega; i \neq a, i \neq b, i \neq c} (1 - 1/\tau_{ai}) ; \quad a \neq b \neq c , \\ \Pi'_a &= \prod_{i=1,2,\dots,\Omega; i \neq a} (1 - 1/\tau_{ai}) , \\ \Pi''_a &= \prod_{i=1,2,\dots,\Omega; i \neq a} (1 - 2/\tau_{ai}) . \end{aligned} \quad (23)$$

With these we can calculate X_{UU} and Y_{UU} ; see [7] for full discussion. For BEGUE(2), the algebra $U(\Omega) \otimes SU(r)$ with $r = 1$ reduces to just $U(\Omega)$ or $SU(\Omega)$. Similarly, f_m is the totally symmetric irrep $\{m\}$ and $f_{m-2} = \{m-2\}$. Therefore to generate f_{m-2} only the action of removal of $\{2\}$ from f_m is allowed. Denoting the last two boxes of f_m by a and a (note that we can remove only boxes from the right end to get a proper Young Tableaux and also boxes in a given row must have the same symbol to apply the results in Table I) as shown in Fig. 2, we have

$$\begin{aligned} \tau_{ai} &= m + i - 1 , \\ \Pi'_a &= \frac{m}{m + \Omega - 1} , \end{aligned} \quad (24)$$

$$\Pi''_a = \frac{m(m-1)}{(m + \Omega - 1)(m + \Omega - 2)} .$$

Similarly $\mathcal{N}_{f_m} = 1$ and $\mathcal{N}_{f_{m-2}} = 1$ as both are symmetric irreps. Now the formulas in Table I will give X_{UU} and then using Eq. (16) we have,

$$\begin{aligned}\mathcal{Q}^{\nu=0}(\{2\}; m, \{m\}) &= \frac{m^2(m-1)^2}{4}, \\ \mathcal{Q}^{\nu=1}(\{2\}; m, \{m\}) &= \frac{m^2(m-1)^2}{4} \frac{2(\Omega+m)(\Omega^2-1)}{m(\Omega+2)}, \\ \mathcal{Q}^{\nu=2}(\{2\}; m, \{m\}) &= \frac{m^2(m-1)^2}{4} \frac{\Omega^2(\Omega-1)(\Omega+m)(\Omega+m+1)}{2(\Omega+2)m(m-1)}.\end{aligned}\tag{25}$$

These and Eq. (15) will give,

$$\langle H^2 \rangle^{\{m\}} = \lambda_{\{2\}}^2 \binom{m}{2} \binom{\Omega+m-1}{2} = \lambda_{\{2\}}^2 \Lambda^{\nu=0}(\Omega, m, 2); \tag{26}$$

$$\Lambda^\nu(\Omega, m, k) = \binom{m-\nu}{k} \binom{\Omega+m+\nu-1}{k}.$$

This agrees with the result stated in [2, 5]. As $P^{\{2\}}(m, \{m\}) = -m(m-1)/2$, we have easily,

$$\hat{\Sigma}_{11}(\{m\}, \{m'\}) = \frac{2\sqrt{m(m-1)(m')(m'-1)}}{\Omega(\Omega+1)\sqrt{(\Omega+m-1)(\Omega+m-2)(\Omega+m'-1)(\Omega+m'-2)}}. \tag{27}$$

Again this agrees, for $m = m'$ with the result stated in [2, 5]. Further, $\hat{\Sigma}_{22}$ is determined only by $X_{\{2\}}$ defined in Eq. (19) and then, using Eq. (25), we have

$$\begin{aligned}\hat{\Sigma}_{22}(\{m\}, \{m'\}) &= \frac{2}{36 \binom{\Omega+2}{3}^2 (\Omega+3) \binom{\Omega+m-1}{2} \binom{\Omega+m'-1}{2}} \\ &\times \left[4\Omega^2(\Omega-1) \binom{\Omega+m+1}{2} \binom{\Omega+m'+1}{2} + 4(\Omega+2)^2(\Omega+3) \binom{m}{2} \binom{m'}{2} \right. \\ &\left. + 4(\Omega^2-1)(\Omega+3)(m-1)(\Omega+m)(m'-1)(\Omega+m') \right].\end{aligned}\tag{28}$$

For $m = m'$, it can be verified that Eq. (28) reduces to

$$\hat{\Sigma}_{22}(\{m\}, \{m'\}) = \frac{2}{(\Omega_m)^2} \sum_{\nu=0}^2 \frac{[\Lambda^\nu(\Omega, m, m-2)]^2 d_\Omega(\mathbf{F}_\nu)}{[\Lambda^{\nu=0}(\Omega, m, 2)]^2}; \quad \Omega_m = \binom{\Omega+m-1}{m} \tag{29}$$

and this agrees with the result given in [5]. Note that \mathbf{F}_ν is $\{0\}$, $\{21^{\Omega-2}\}$ and $\{42^{\Omega-2}\}$ for $\nu = 0, 1$ and 2 respectively. It is useful to mention that Eqs. (27) and (28) follow from the

results for fermion systems given in [13] with $\Omega \rightarrow -\Omega$ symmetry. Finally, it is useful to mention that in the $m \rightarrow \infty$ and Ω finite limit we have,

$$\begin{aligned}\hat{\Sigma}_{11}(\{m\}, \{m'\}) &= \frac{2}{\Omega(\Omega+1)} , \\ \hat{\Sigma}_{22}(\{m\}, \{m'\}) &= 8 \frac{\Omega^2(\Omega-1) + (\Omega+2)^2(\Omega+3) + 4(\Omega^2-1)(\Omega+3)}{\Omega^2(\Omega+1)^2(\Omega+2)^2(\Omega+3)} .\end{aligned}\tag{30}$$

Non-vanishing of $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{22}$ for finite Ω in the $m \rightarrow \infty$ limit is interpreted in [5, 14] as non-ergodicity of BEGUE ensembles. See the discussion in [15] for the resolution of this problem.

IV. EMBEDDED GAUSSIAN UNITARY ENSEMBLE FOR BOSONS WITH F -SPIN: BEGUE(2)-SU(2) WITH $r = 2$

For two species boson systems we have BEGUE(2)- $SU(2)$ and then the formulation in Section 2 with $r = 2$ will be applicable. Here the two species are assumed to be the two components of a fictitious F -spin as discussed recently in [8]. For such a m boson system, the $SU(\Omega)$ irreps will be two rowed denoted by $f_m = \{m-r, r\}$ with $F = \frac{m}{2} - r$. With this, there are three allowed f_{m-2} irreps as shown in Fig. 3. The irreps in (i) and (iii) in the figure can be obtained by removing $f_2 = \{2\}$ from f_m . However for (ii) in the figure both $\{2\}$ and $\{1^2\}$ will apply. For $f_{m-2} = \{m-r-2, r\}$ irrep [this corresponds to (i) in Fig. 3] we have

$$\begin{aligned}\tau_{a2} &= m - 2r + 1 , \\ \tau_{ai} &= m - r + i - 1 ; i = 3, 4, \dots, \Omega , \\ \Pi'_a &= \frac{(m-2r)(m-r+1)}{(m-2r+1)(m-r+\Omega-1)} , \\ \Pi''_a &= \frac{(m-2r-1)(m-r)(m-r+1)}{(m-2r+1)(m-r+\Omega-1)(m-r+\Omega-2)} .\end{aligned}\tag{31}$$

Similarly for $f_{m-2} = \{m-r, r-2\}$ irrep [this corresponds to (iii) in Fig. 3] we have

$$\begin{aligned}\tau_{b1} &= 2r - m - 1 , \\ \tau_{bi} &= r + i - 2 , i = 3, 4, \dots, \Omega \\ \Pi'_b &= \frac{(r)(2r-m-2)}{(2r-m-1)(r+\Omega-2)} , \\ \Pi''_b &= \frac{(2r-m-3)(r)(r-1)}{(2r-m-1)(r+\Omega-2)(r+\Omega-3)} .\end{aligned}\tag{32}$$

TABLE I. Formulas for $X_{UU}(f_2; f_{m-2}, f'_{m-2}; \mathbf{F}_\nu)$ and $Y_{UU}(f_{m-2}, f'_{m-2}; \mathbf{F}_\nu)$ with $\nu = 1, 2$.

$\{f_{m-2}\} \{f'_{m-2}\}$	$X_{UU}(\{1^2\}; f_{m-2}, f'_{m-2}; \{2^\nu, 1^{\Omega-2\nu}\})$
$\{f(ab)\} \{f(ab)\}$	$\frac{\Omega}{(\Omega-2)} \left\{ \delta_{\nu,2} + \frac{(\Omega-1)(\Omega-2)}{2\Pi_a^{(b)}\Pi_b^{(a)}} \delta_{\nu,2} + (3-2\nu) \frac{(\Omega-1)}{2} \right.$ $\times \left[\left(1 + \frac{1}{\tau_{ab}}\right) \frac{1}{\Pi_b^{(a)}} + \left(1 - \frac{1}{\tau_{ab}}\right) \frac{1}{\Pi_a^{(b)}} - \frac{4}{\Omega} \delta_{\nu,1} \right] \left. \right\}$
$\{f(ab)\} \{f(ac)\}$	$\frac{\Omega(\Omega-1)}{2(\Omega-2)} \left\{ \frac{2}{(\Omega-1)} \delta_{\nu,2} - \frac{4}{\Omega} \delta_{\nu,1} + (3-2\nu) \frac{1}{\Pi_a^{(bc)}} \right\}$
$\{f_{m-2}\} \{f'_{m-2}\}$	$X_{UU}(\{2\}; f_{m-2}, f'_{m-2}; \{2\nu, \nu^{\Omega-2}\})$
$\{f(ab)\} \{f(ab)\}$	$\frac{\Omega(\Omega+1)}{2} \left\{ \frac{1}{\Pi_a^{(b)}\Pi_b^{(a)}} \delta_{\nu,2} + \frac{2}{(\Omega+1)(\Omega+2)} \delta_{\nu,2} \right.$ $+ (3-2\nu) \frac{1}{(\Omega+2)} \left[\frac{(\tau_{ab}-1)^2}{\tau_{ab}(\tau_{ab}+1)} \frac{1}{\Pi_b^{(a)}} + \frac{(\tau_{ab}+1)^2}{\tau_{ab}(\tau_{ab}-1)} \frac{1}{\Pi_a^{(b)}} - \frac{4}{\Omega} \delta_{\nu,1} \right] \left. \right\}$
$\{f(aa)\} \{f(aa)\}$	$\frac{\Omega}{(\Omega+2)} \left\{ \delta_{\nu,2} + (3-2\nu) \frac{2(\Omega+1)}{\Pi_a'} + \frac{(\Omega+1)(\Omega+2)}{2\Pi_a''} \delta_{\nu,2} - \frac{2(\Omega+1)}{\Omega} \delta_{\nu,1} \right\}$
$\{f(aa)\} \{f(bb)\}$	$-\frac{2(\Omega+1)}{(\Omega+1)} \delta_{\nu,1} + \frac{\Omega}{(\Omega+2)} \delta_{\nu,2}$
$\{f(aa)\} \{f(ab)\}$	$\frac{\Omega}{(\Omega+2)} \left\{ \delta_{\nu,2} + (3-2\nu) \frac{(\Omega+1)(\tau_{ab}+1)}{(\tau_{ab}-1)\Pi_a^{(b)}} - \frac{2(\Omega+1)}{\Omega} \delta_{\nu,1} \right\}$
$\{f_{m-2}\} \{f'_{m-2}\}$	$Y_{UU}(f_{m-2}, f'_{m-2}; \{2, 1^{\Omega-2}\})$
$\{f(ab)\} \{f(ab)\}$	$-\frac{\Omega}{2} \left[\frac{(\Omega^2-1)}{(\Omega^2-4)} \right]^{1/2} \left\{ \left(1 + \frac{1}{\tau_{ab}}\right) \frac{1}{\Pi_a^{(b)}} + \left(1 - \frac{1}{\tau_{ab}}\right) \frac{1}{\Pi_b^{(a)}} - \frac{4}{\Omega} \right\}$
$\{f(ab)\} \{f(ac)\}$	$-\frac{\Omega}{2} \left[\frac{(\Omega^2-1)}{(\Omega^2-4)} \right]^{1/2} \left\{ \left(1 + \frac{1}{\tau_{ac}}\right) \frac{1}{\Pi_a^{(b)}} - \frac{4}{\Omega} \right\}$
$\{f(ab)\} \{f(aa)\}$	$-\Omega \left[\frac{(\Omega^2-1)}{(\Omega^2-4)} \right]^{1/2} \left\{ \frac{1}{\Pi_a^{(b)}} - \frac{2}{\Omega} \right\}$

Finally, for $f_{m-2} = \{m-r-1, r-1\}$ irrep [this corresponds to (ii) in Fig. 3] we have

$$\begin{aligned}
 \tau_{ab} &= m - 2r + 1 = 2F + 1, \\
 \tau_{ai} &= m - r + i - 1, \quad \tau_{bi} = r + i - 2; \quad i = 3, 4, \dots, \Omega, \\
 \Pi_a^{(b)} &= \frac{(m-r+1)}{(m-r+\Omega-1)}, \\
 \Pi_b^{(a)} &= \frac{(r)}{(r+\Omega-2)}.
 \end{aligned} \tag{33}$$

These and $\mathcal{N}_{f_{m-2}}/\mathcal{N}_{f_m}$ will give the formulas for the lower order moments of one and two point functions as described in Section 2. The dimension ratios needed are,

$$\begin{aligned}\frac{\mathcal{N}_{\{m-r-2,r\}}}{\mathcal{N}_{\{m-r,r\}}} &= \frac{(m-r)(m-r+1)(m-2r-1)}{m(m-1)(m-2r+1)}, \\ \frac{\mathcal{N}_{\{m-r-1,r-1\}}}{\mathcal{N}_{\{m-r,r\}}} &= \frac{r(m-r+1)}{m(m-1)}, \\ \frac{\mathcal{N}_{\{m-r,r-2\}}}{\mathcal{N}_{\{m-r,r\}}} &= \frac{r(r-1)(m-2r+3)}{m(m-1)(m-2r+1)}.\end{aligned}\tag{34}$$

Using Eqs. (31)-(34) and the expressions in Table I, it is possible to derive analytical formulas for the P 's, \mathcal{Q} 's and \mathcal{R} 's that define $\langle H^2 \rangle$, $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{22}$. The final formulas (obtained using MATHEMATICA) are, with (m, F) defining f_m ,

$$\begin{aligned}P^{\{2\}}(m, F) &= -\frac{1}{8} [3m(m-2) + 4F(F+1)] , \\ P^{\{1^2\}}(m, F) &= -\frac{1}{8} [m(m+2) - 4F(F+1)] , \\ \mathcal{Q}^{\nu=0}(\{2\} : m, F) &= [P^{\{2\}}(m, F)]^2 , \\ \mathcal{Q}^{\nu=0}(\{1^2\} : m, F) &= [P^{\{1^2\}}(m, F)]^2 , \\ \mathcal{Q}^{\nu=1}(\{2\} : m, F) &= \frac{(\Omega+1)}{16(\Omega+2)} \\ &\times [2(\Omega-2)P^{\{2\}}(m, F) \{3(2\Omega+m)(m-2) + 4F(F+1)\} + 8\Omega(m-1)(\Omega+2m-4)F(F+1)] , \\ \mathcal{Q}^{\nu=1}(\{1^2\} : m, F) &= \frac{(\Omega-1)P^{\{1^2\}}(m, F)}{8} [(2\Omega+m)(m+2) - 4F(F+1)] , \\ \mathcal{Q}^{\nu=2}(\{2\} : m, F) &= \frac{(\Omega)}{8(\Omega+2)} [(3\Omega^2 + 7\Omega + 6)[F(F+1)]^2 \\ &+ \frac{3}{16} m(m-2)(2\Omega+m)(2\Omega+m+2)(\Omega-1)(\Omega-2) \\ &+ \frac{F(F+1)}{2} \{m(2\Omega+m)(5\Omega+3)(\Omega-2) + 2\Omega(\Omega^2-1)(\Omega-6)\}] , \\ \mathcal{Q}^{\nu=2}(\{1^2\} : m, F) &= \frac{\Omega(\Omega-3)P^{\{1^2\}}(m, F)}{16} [(2\Omega+m)(2\Omega+m-2) - 4F(F+1)] , \\ \mathcal{R}^{\nu=0}(m, F) &= P^{\{2\}}(m, F) P^{\{1^2\}}(m, F) , \\ \mathcal{R}^{\nu=1}(m, F) &= \sqrt{\frac{\Omega^2-1}{\Omega^2-4}} \frac{(2-\Omega)P^{\{1^2\}}(m, F)}{8} \{4[F(F+1) - 3\Omega] + 3m(2\Omega+m-2)\} .\end{aligned}\tag{35}$$

Note that Eq. (35) is related to the EGUE(2)- $SU(2)$ results given in [6] by the $\Omega \rightarrow -\Omega$ transformation and they are also closely related to the results for spectral variances given in [8].

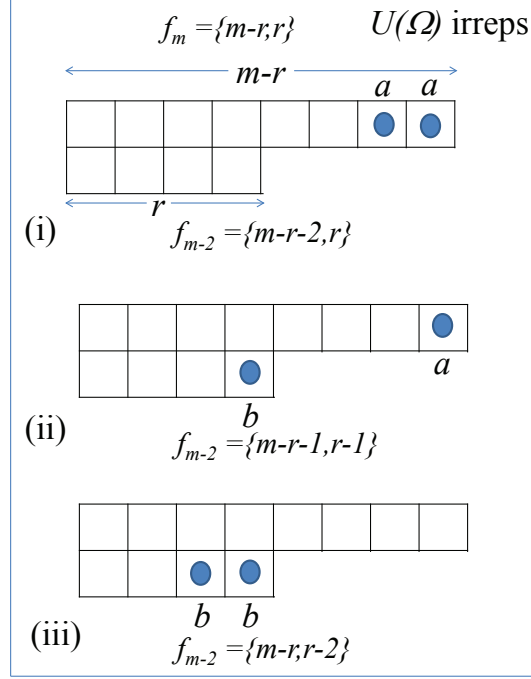


FIG. 3. Young tableaux denoting the two-rowed $SU(\Omega)$ irreps $f_m = \{m - r, r\}$ appropriate for BEGUE(2)- $SU(2)$. Removal of two boxes generating $m - 2$ particle irreps f_{m-2} are also shown in the figure. For (ii) both the irreps $f_2 = \{2\}$ and $\{1^2\}$ will apply while for (ii) and (iii) only $\{2\}$ will apply.

V. EMBEDDED GAUSSIAN UNITARY ENSEMBLE FOR SPIN ONE BOSONS: BEGUE(2)- $SU(3)$ WITH $r = 3$

Spin one boson systems, discussed in [9], possess $U(3\Omega) \supset U(\Omega) \otimes [SU(3) \supset SO(3)]$ symmetry. Instead of BEGOE(2) or BEGUE(2) generated by random two-body interactions preserving total spin S , it is also possible to consider interactions preserving the $SU(3)$ symmetry. Then, for the GUE version, we have BEGUE(2)- $SU(3)$ that corresponds to $r = 3$ in Section 2. As $U(3)$ irreps will have, in young tableaux representation, maximum 3 rows, the $U(\Omega)$ irrep also will have maximum three rows. Given m bosons in Ω number of sp levels, the allowed $U(\Omega)$ irreps are $\{f_1, f_2, f_3, f_4, \dots, f_\Omega\}$ with $f_1 + f_2 + f_3 = m$, $f_1 \geq f_2 \geq f_3 \geq 0$ and $f_i = 0$ for $i = 4, 5, \dots, \Omega$. Because of the last condition we use simply $\{f_1, f_2, f_3\}$. For $f_2 = 0$ and $f_3 = 0$, we have totally symmetric irreps with $\{f_1\} = \{m\}$ and then all the results

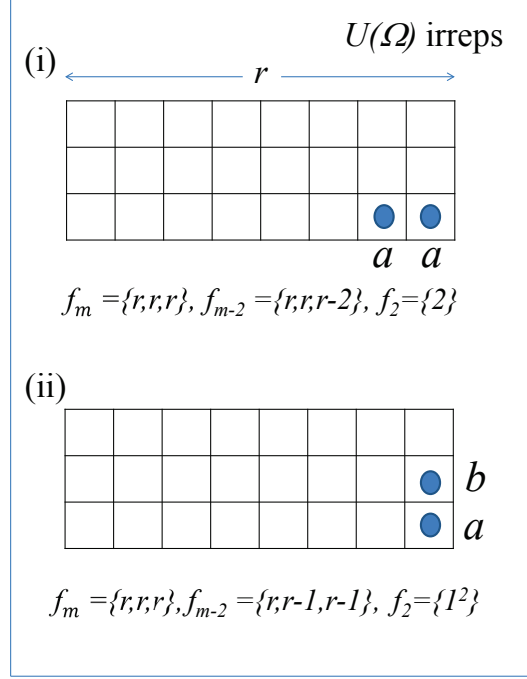


FIG. 4. Young tableaux denoting the three-column $SU(\Omega)$ irreps $f_m = \{r, r, r\}$, $m = 3r$ appropriate for BEGUE(2)- $SU(2)$. Removal of two boxes generating $m - 2$ particle irreps f_{m-2} are also shown in the figure. For (i) only the irrep $f_2 = \{2\}$ will apply while for (ii) only $\{1^2\}$ will apply.

derived in Section 3 will apply directly. Similarly, for $f_2 \neq 0$ and $f_3 = 0$, all the results of Section 4 will apply. Thus the non-trivial irreps for BEGUE(2)- $SU(3)$ are the m -boson irreps $f_m = \{f_1, f_2, f_3\}$ with $f_3 \neq 0$. Given a f_m in general there will be six f_{m-2} and they are $\{f_1 - 2, f_2, f_3\}$, $\{f_1, f_2 - 2, f_3\}$, $\{f_1, f_2, f_3 - 2\}$, $\{f_1 - 1, f_2 - 1, f_3\}$, $\{f_1 - 1, f_2, f_3 - 1\}$, $\{f_1, f_2 - 1, f_3 - 1\}$. Therefore, as seen from Section 2, deriving analytical formulas for P 's, \mathcal{Q} 's and \mathcal{R} 's that determine $\langle H^2 \rangle$, $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{22}$ will be cumbersome. One situation that is amenable to analytical treatment is for the irreps $\{r, r, r\}$, $m = 3r$. For this class of irreps, the f_{m-2} are simple as shown in Fig. 4. For $f_{m-2} = \{r, r, r - 2\}$ we need Π'_a and Π''_a and they are given by,

$$\Pi'_a = \frac{3r}{\Omega + r - 3}, \quad \Pi''_a = \frac{6r(r - 1)}{(\Omega + r - 3)(\Omega + r - 4)}. \quad (36)$$

Similarly, for $f_{m-2} = \{r, r - 1, r - 1\}$ we need τ_{ab} , $\Pi_a^{(b)}$ and $\Pi_b^{(a)}$ and they are,

$$\tau_{ab} = -1, \quad \Pi_a^{(b)} = \frac{3r}{2(\Omega + r - 3)}, \quad \Pi_b^{(a)} = \frac{2(r + 1)}{(\Omega + r - 2)}. \quad (37)$$

$$\lambda_{\{2\}}^2 = \lambda_{\{1^2\}}^2 = 1 \quad \Omega=6$$

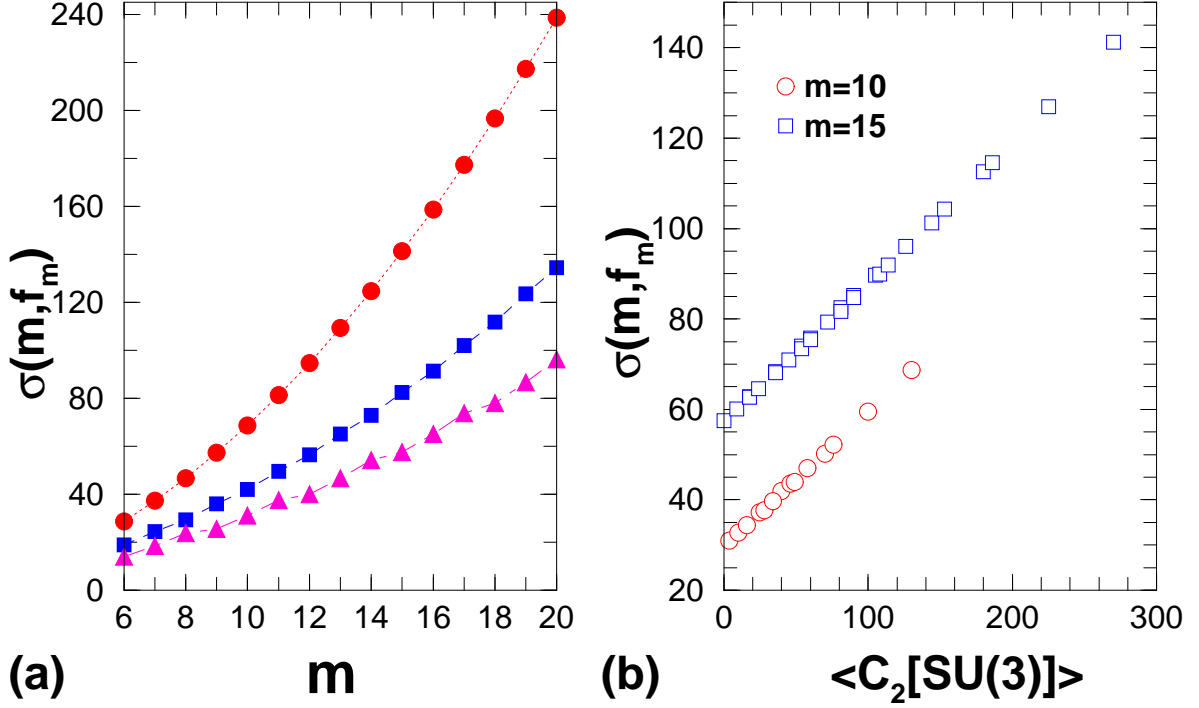


FIG. 5. (a) Variation of spectral widths as a function of m with fixed f_m . Shown are the results for f_m as one-rowed $f_m^{(k)}$ irreps (red circles), two-rowed $f_m^{(k)}$ irreps (blue squares) and three-rowed $f_m^{(k)}$ irreps (magenta triangles). (b) Variation of spectral widths as a function of f_m with fixed m . Shown are the results for $\Omega = 6$ and $m = 10, 15$. Instead of showing f_m , we have used $\langle C_2[SU(3)] \rangle^{\tilde{f}_m}$.

In addition, ratio of the S_Ω dimensions needed are,

$$\frac{\mathcal{N}_{r,r,r-2}}{\mathcal{N}_{r,r,r}} = \frac{2(r-1)}{(3r-1)}, \quad \frac{\mathcal{N}_{r,r-1,r-1}}{\mathcal{N}_{r,r,r}} = \frac{r+1}{(3r-1)}. \quad (38)$$

With these, carrying out simplification of the formulas given in Table I will give the following results,

$$\begin{aligned}
P^{\{2\}}(m, \{r, r, r\}) &= -3r(r-1) , \quad P^{\{1^2\}}(m, \{r, r, r\}) = -\frac{3}{2}r(r+1) , \\
\mathcal{Q}^{\nu=0}(\{2\} : m, \{r, r, r\}) &= (3r)^2(r-1)^2 , \\
\mathcal{Q}^{\nu=0}(\{1^2\} : m, \{r, r, r\}) &= \frac{(3r)^2(r+1)^2}{4} , \\
\mathcal{Q}^{\nu=1}(\{2\} : m, \{r, r, r\}) &= \frac{6(\Omega+1)(\Omega-3)r(r-1)^2(\Omega+r)}{(\Omega+2)} , \\
\mathcal{Q}^{\nu=1}(\{1^2\} : m, \{r, r, r\}) &= \frac{3(\Omega-1)(\Omega-3)r(r+1)^2(\Omega+r)}{2(\Omega-2)} , \\
\mathcal{Q}^{\nu=2}(\{2\} : m, \{r, r, r\}) &= \frac{3\Omega(\Omega-2)(\Omega-3)r(r-1)(\Omega+r)(\Omega+r+1)}{4(\Omega+2)} , \\
\mathcal{Q}^{\nu=2}(\{1^2\} : m, \{r, r, r\}) &= \frac{3\Omega(\Omega-3)(\Omega-4)r(r+1)(\Omega+r)(\Omega+r-1)}{8(\Omega-2)} , \\
\mathcal{R}^{\nu=0}(m, \{r, r, r\}) &= \frac{(3r)^2(r^2-1)}{2} , \\
\mathcal{R}^{\nu=1}(m, \{r, r, r\}) &= -\sqrt{\frac{\Omega^2-1}{\Omega^2-4}} 3(\Omega-3)r(r^2-1)(\Omega+r) .
\end{aligned} \tag{39}$$

Using these equations one can calculate the variances $\langle H^2 \rangle$ and the covariances $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{22}$ for irreps of the type $\{r, r, r\}$. For example, Eq. (15) can be simplified using Eq. (39) to give a compact formula for spectral variances,

$$\begin{aligned}
\overline{\langle H^2 \rangle^{m, \{r, r, r\}}} &= \lambda_{\{2\}}^2 \left[\frac{3}{2}r(r-1)(\Omega+r-3)(\Omega+r-4) \right] \\
&+ \lambda_{\{1^2\}}^2 \left[\frac{3}{4}r(r+1)(\Omega+r-2)(\Omega+r-3) \right] .
\end{aligned} \tag{40}$$

It is also possible to derive analytical results for the irreps $f_m = \{r+1, r, r\}$ and $\{r+2, r, r\}$ just as it was done for $\{4^r, p\}$ irreps for EGUE(2)- $SU(4)$ ensemble in [7]. The results are as follows. For these irreps, the allowed f_{m-2} irreps and the corresponding τ and Π_{\dots} functions are given in Table II and the dimension ratios in Table III. Using these, for $f_m = \{r+1, r, r\}$

$$\Omega=6$$

$$\lambda_{\{2\}}^2 = \lambda_{\{1\}}^2 = 1$$

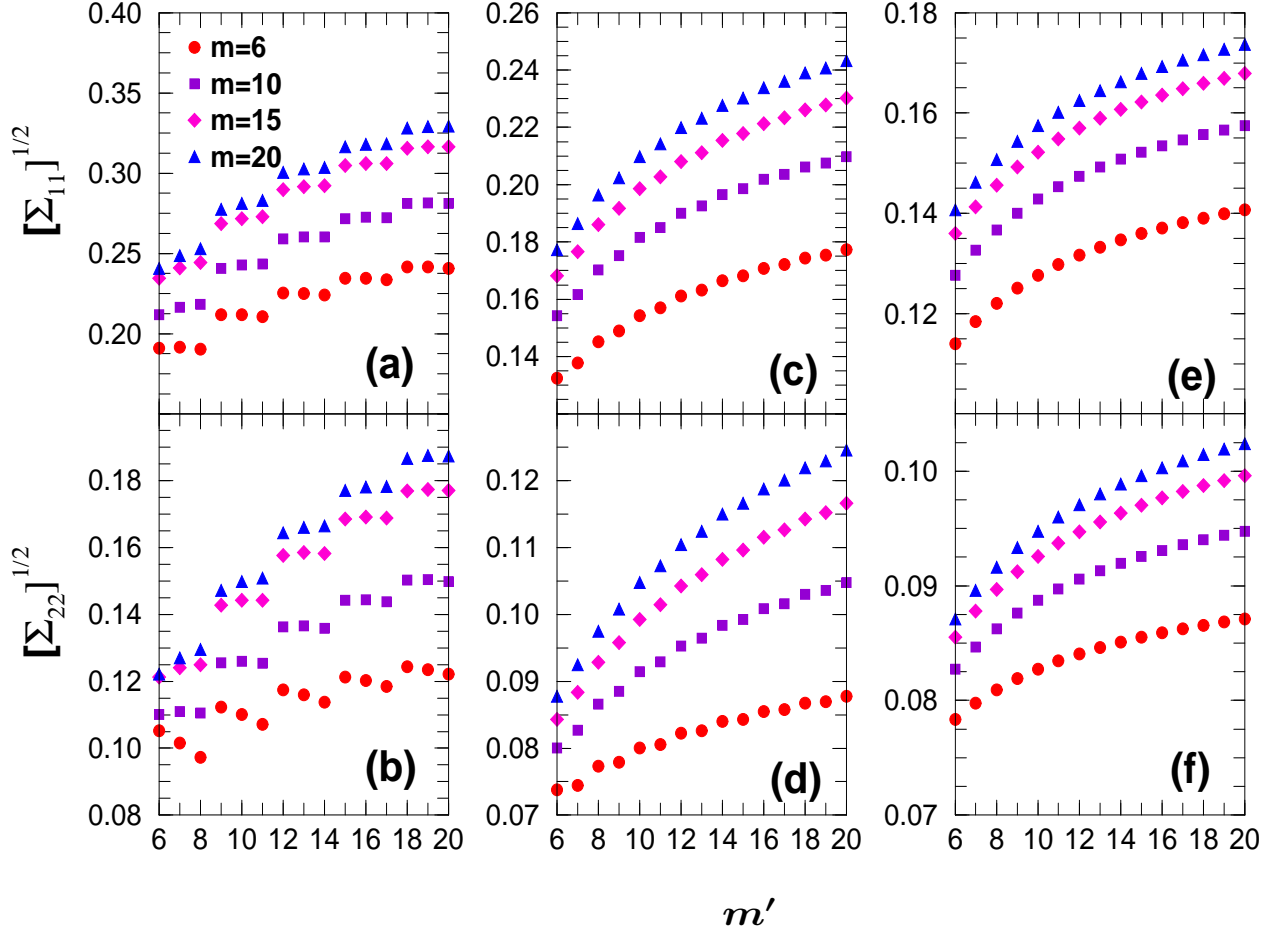


FIG. 6. Self and cross correlations in energy centroids and spectral variances as a function of m and m' for $\Omega = 6$ (with fixed f_m and $f_{m'}$): (a) $[\Sigma_{11}(m, f_m; m', f_{m'})]^{1/2}$ with $\{f_m\} = \{m/3, m/3, m/3\}$ for $m \bmod 3 = 0$, $\{f_m\} = \{(m+2)/3, (m-1)/3, (m-1)/3\}$ for $m \bmod 3 = 1$ and $\{f_m\} = \{(m+4)/3, (m-2)/3, (m-2)/3\}$ for $m \bmod 3 = 2$ and similarly $f_{m'}$ is defined; (b) $[\Sigma_{22}(m, f_m; m', f_{m'})]^{1/2}$ with $\{f_m\} = \{m/3, m/3, m/3\}$ for $m \bmod 3 = 0$, $\{f_m\} = \{(m+2)/3, (m-1)/3, (m-1)/3\}$ for $m \bmod 3 = 1$ and $\{f_m\} = \{(m+4)/3, (m-2)/3, (m-2)/3\}$ for $m \bmod 3 = 2$ and similarly $f_{m'}$ is defined; (c) $[\Sigma_{11}(m, f_m; m', f_{m'})]^{1/2}$ with $\{f_m\} = \{m/2, m/2\}$ for $m \bmod 2 = 0$ and $\{f_m\} = \{(m+1)/2, (m-1)/2\}$ for $m \bmod 2 = 1$ and similarly $f_{m'}$ is defined; (d) $[\Sigma_{22}(m, f_m; m', f_{m'})]^{1/2}$ with $\{f_m\} = \{m/2, m/2\}$ for $m \bmod 2 = 0$ and $\{f_m\} = \{(m+1)/2, (m-1)/2\}$ for $m \bmod 2 = 1$ and similarly $f_{m'}$ is defined; (e) $[\Sigma_{11}(m, f_m; m', f_{m'})]^{1/2}$ with $\{f_m\} = \{m\}$ and $\{f_{m'}\} = \{m'\}$; (f) $[\Sigma_{22}(m, f_m; m', f_{m'})]^{1/2}$ with $\{f_m\} = \{m\}$ and $\{f_{m'}\} = \{m'\}$.

$$\Omega = 6$$

$$\lambda^2_{\{2\}} = \lambda^2_{\{1^2\}} = 1$$

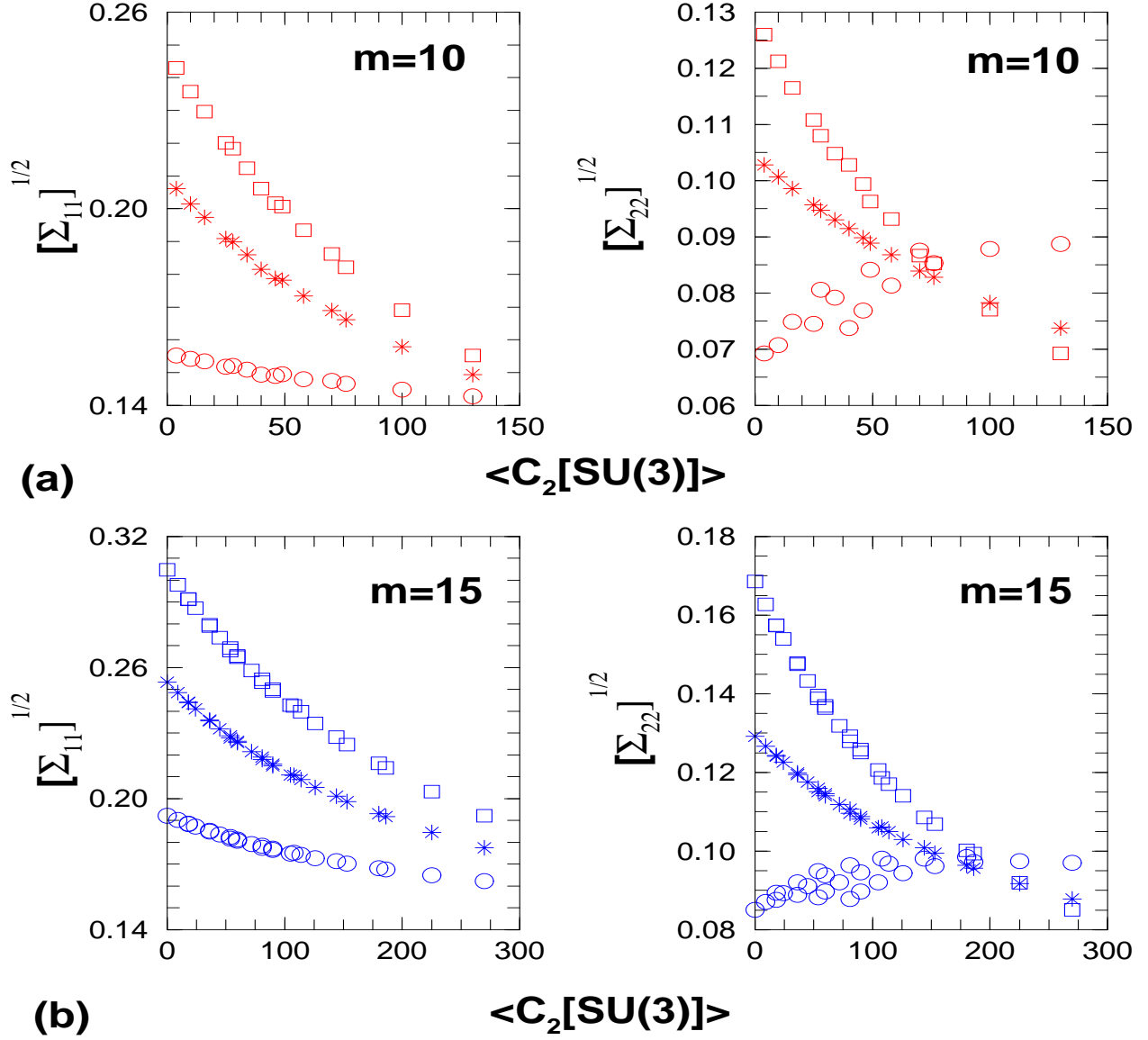


FIG. 7. Self and cross correlations in energy centroids and spectral variances as a function of f_m and $f_{m'}$ for $\Omega = 6$ (with fixed $m = m'$). Results are shown for (a) $m = m' = 10$ with $f_m = \{10\}$ (red circles), $\{5, 5\}$ (red stars) and $\{4, 3, 3\}$ (red squares) with all one, two and three rowed $f_{m'}$ irreps; (b) $m = m' = 15$ with $f_m = \{15\}$ (blue circles), $\{8, 7\}$ (blue stars) and $\{5, 5, 5\}$ (blue squares) with all one, two and three rowed $f_{m'}$ irreps.

irreps, P , \mathcal{Q} and \mathcal{R} functions are,

$$\begin{aligned}
P^{\{2\}}(m, \{r+1, r, r\}) &= -r(3r-1) , \quad P^{\{1^2\}}(m, \{r+1, r, r\}) = -\frac{r}{2}(5+3r) , \\
\mathcal{Q}^{\nu=0}(\{2\} : m, \{r+1, r, r\}) &= r^2(3r-1)^2 , \\
\mathcal{Q}^{\nu=0}(\{1^2\} : m, \{r+1, r, r\}) &= \frac{r^2}{4}(5+3r)^2 , \\
\mathcal{Q}^{\nu=1}(\{2\} : m, \{r+1, r, r\}) &= \frac{r(1+\Omega)}{2+\Omega} [6r^3(-3+\Omega) + 3(-3+\Omega)\Omega \\
&\quad + 2r^2(-3+\Omega)(-2+3\Omega) + r\{-2+3(9-2\Omega)\Omega\}] , \\
\mathcal{Q}^{\nu=1}(\{1^2\} : m, \{r+1, r, r\}) &= \frac{r(-1+\Omega)}{2(-2+\Omega)} [-r(5+3r)^2 \\
&\quad + \{-18+r(-18+r+3r^2)\}\Omega + 3(1+r)(2+r)\Omega^2] , \\
\mathcal{Q}^{\nu=2}(\{2\} : m, \{r+1, r, r\}) &= \frac{r(-3+\Omega)\Omega(1+r+\Omega)}{4(2+\Omega)} [8+3r^2(-2+\Omega) \\
&\quad - (-8+\Omega)\Omega + r(-2+\Omega)(1+3\Omega)] , \\
\mathcal{Q}^{\nu=2}(\{1^2\} : m, \{r+1, r, r\}) &= \frac{r(-3+\Omega)\Omega(-1+r+\Omega)}{8(-2+\Omega)} [-16+3r^2(-4+\Omega) \\
&\quad + r(-4+\Omega)(7+3\Omega) + \Omega(-14+5\Omega)] , \\
\mathcal{R}^{\nu=0}(m, \{r+1, r, r\}) &= \frac{r^2}{2}(-1+3r)(5+3r) , \\
\mathcal{R}^{\nu=1}(m, \{r+1, r, r\}) &= \frac{r}{24}\sqrt{\frac{\Omega^2-1}{\Omega^2-4}} [r^3(468-151\Omega) + 153(-3+\Omega)\Omega \\
&\quad + r^2\{600+(283-151\Omega)\Omega\} - 5r\{60+13\Omega(-7+2\Omega)\}] .
\end{aligned} \tag{41}$$

Similarly, for $f_m = \{r+2, r, r\}$ irreps we have,

$$\begin{aligned}
P^{\{2\}}(m, \{r+2, r, r\}) &= -(3r^2+r+1) , \quad P^{\{1^2\}}(m, \{r+2, r, r\}) = -\frac{r}{2}(7+3r) , \\
\mathcal{Q}^{\nu=0}(\{2\} : m, \{r+2, r, r\}) &= (3r^2+r+1)^2 , \\
\mathcal{Q}^{\nu=0}(\{1^2\} : m, \{r+2, r, r\}) &= \frac{r^2}{4}(7+3r)^2 , \\
\mathcal{Q}^{\nu=1}(\{2\} : m, \{r+2, r, r\}) &= \frac{r(1+\Omega)}{2(2+\Omega)} [-4(1+r+3r^2)^2 \\
&\quad + \{2+r\{3+r\{51+4r(-7+3r)\}\}\}\Omega + (2+11r+12r^3)\Omega^2] , \\
\mathcal{Q}^{\nu=1}(\{1^2\} : m, \{r+2, r, r\}) &= \frac{r(-1+\Omega)}{2(-2+\Omega)} [-r(7+3r)^2 \\
&\quad + (5+3r)(-6+r^2)\Omega + \{10+3r(4+r)\}\Omega^2] , \\
\mathcal{Q}^{\nu=2}(\{2\} : m, \{r+2, r, r\}) &= \frac{\Omega}{4(2+\Omega)} [3r^4(-3+\Omega)(-2+\Omega) + (-1+\Omega)\Omega(2+\Omega)(3+\Omega) \\
&\quad + 2r^3(-3+\Omega)(-2+\Omega)(4+3\Omega) + r^2(-3+\Omega)\Omega\{7+3\Omega(1+\Omega)\} \\
&\quad + r(-3+\Omega)\{22+\Omega\{42+\Omega(19+\Omega)\}\}] ,
\end{aligned} \tag{42}$$

TABLE II. Formulas for the functions $\Pi_{--}^{(-)}$'s defined in Eq. (23) as required for $\{f_m\}$ irreps $\{r+1, r, r\}$ and $\{r+2, r, r\}$. Given also are the values of axial distances (τ_{--} 's). Also, $\Pi_a^{(bc)} = r/(\Omega + r - 3)$ for both $\{f_m\}$ examples shown in the Table.

$\{f_m\}$	$\{f_{m-2}\}$	Required functions
$\{r+1, r, r\}$	$\{r+1, r, r-2\}$	$\Pi'_a = \frac{8r}{3(\Omega + r - 3)}, \Pi''_a = \frac{5r(r-1)}{(\Omega + r - 3)(\Omega + r - 4)}$
$f(aa)$		
$\{r+1, r-1, r-1\}$	$\tau_{ab} = -1, \Pi_a^{(b)} = \frac{4r}{3(\Omega + r - 3)}, \Pi_b^{(a)} = \frac{3(r+1)}{2(\Omega + r - 2)}$	
$f(ab)$		
$\{r, r, r-1\}$	$\tau_{ac} = -3, \Pi_a^{(c)} = \frac{2r}{\Omega + r - 3}, \Pi_c^{(a)} = \frac{r+3}{2(\Omega + r)}$	
$f(ac)$		
$\{r+2, r, r\}$	$\{r+2, r, r-2\}$	$\Pi'_a = \frac{5r}{2(\Omega + r - 3)}, \Pi''_a = \frac{9r(r-1)}{2(\Omega + r - 3)(\Omega + r - 4)}$
$f(aa)$		
$\{r+2, r-1, r-1\}$	$\tau_{ab} = -1, \Pi_a^{(b)} = \frac{5r}{4(\Omega + r - 3)}, \Pi_b^{(a)} = \frac{4(r+1)}{3(\Omega + r - 2)}$	
$f(ab)$		
$\{r+1, r, r-1\}$	$\tau_{ac} = -4, \Pi_a^{(c)} = \frac{2r}{\Omega + r - 3}, \Pi_c^{(a)} = \frac{2(r+4)}{3(\Omega + r + 1)}$	
$f(ac)$		
$\{r, r, r\}$	$\Pi'_c = \frac{r+4}{2(\Omega + r + 1)}, \Pi''_c = \frac{(r+3)(r+4)}{6(\Omega + r)(\Omega + r + 1)}$	
$f(cc)$		

$$\begin{aligned}
\mathcal{Q}^{\nu=2}(\{1^2\} : m, \{r+2, r, r\}) &= \frac{r(-3+\Omega)\Omega(-1+r+\Omega)}{8(-2+\Omega)} [-44 + 3r^2(-4+\Omega) \\
&\quad + r(-4+\Omega)(11+3\Omega) + \Omega(-12+7\Omega)] , \\
\mathcal{R}^{\nu=0}(m, \{r+2, r, r\}) &= \frac{r}{2}(7+3r)(1+r+3r^2) , \\
\mathcal{R}^{\nu=1}(m, \{r+2, r, r\}) &= \frac{r}{48}\sqrt{\frac{\Omega^2-1}{\Omega^2-4}} [8(7+3r)(12+r+37r^2) \\
&\quad + \{-1008 + r\{1106 + (241 - 289r)r\}\}\Omega + \{208 - r(489 + 289r)\}\Omega^2] .
\end{aligned}$$

In addition to analytical results, as stated in Section 2, one can use the tables in [7] (also to a large extent Table I) and obtain numerical results for the variation of spectral variances with the eigenvalues of the quadratic Casimir invariant of $U(\Omega)$ or equivalently $C_2[SU(3)]$, for various (Ω, m) values and also for both self and cross correlations in energy centroids

TABLE III. Dimension ratios with respect to the S_m group for the examples in Table II.

$\{f_m\}$	$\{f_{m-2}\}$	$\frac{\mathcal{N}_{\{f_{m-2}\}}}{\mathcal{N}_{\{f_m\}}}$
$\{r+1, r, r\}$	$\{r+1, r, r-2\}$	$\frac{5(r-1)}{3(3r+1)}$
	$\{r+1, r-1, r-1\}$	$\frac{2(r+1)}{3(3r+1)}$
	$\{r, r, r-1\}$	$\frac{(r+3)}{3(3r+1)}$
$\{r+2, r, r\}$	$\{r+2, r, r-2\}$	$\frac{9r(r-1)}{2(3r+2)(3r+1)}$
	$\{r+2, r-1, r-1\}$	$\frac{5r(r+1)}{3(3r+2)(3r+1)}$
	$\{r+1, r, r-1\}$	$\frac{4r(r+4)}{3(3r+2)(3r+1)}$
	$\{r, r, r\}$	$\frac{(r+3)(r+4)}{6(3r+2)(3r+1)}$

and spectral variances. Note that For a $\Omega = 6$ system with $\lambda_{\{2\}}^2 = \lambda_{\{1^2\}} = 1$ calculations are carried out for various choices of m and f_m and the results are shown in 5, 6 and 7. Let us mention that $C_2[SU(3)]$ for a irrep $\{f_1, f_2, f_3\}$ is given by the formula,

$$\langle C_2[SU(3)] \rangle^{\{f_1, f_2, f_3\}} = \lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu) ; \quad \lambda = f_1 - f_2, \mu = f_2 - f_3 . \quad (43)$$

It is seen from Fig. 5a that the spectral widths will be largest for one rowed irreps and smallest for three row irreps for a fixed m . Also, widths as expected increase with m . Similarly, Fig. 5b shows that for a fixed m , widths increase as the eigenvalue of $C_2[SU(3)]$ increases and this is consistent with the observation in Fig. 5a as the eigenvalue of $C_2[SU(3)]$ is largest for totally symmetric irrep. Results in Fig. 6 show that: (i) the centroid and variance fluctuations increase with m' for fixed m and vice-versa; (ii) they are larger for three rowed irreps compared to those for one rowed irreps; (iii) centroid fluctuations are much larger than variance fluctuations as seen before also for EGUE(2)-s and EGUE(2)- $SU(4)$ ensembles. Similar trends are also seen for $m = m'$ but varying $f_{m'}$ with fixed f_m and these results are shown in Fig. 7. A different trend is seen for the covariances in spectral variances for the totally symmetric irrep $f_m = \{m\}$ and $f_{m'}$ varying. More importantly, the centroid and variance fluctuations are smallest for the ground state i.e., the most symmetric irrep for bosons. It is seen from Figs. 6 and 7 that the covariances in energy centroids are $\sim 15 - 25\%$ and the covariances in spectral variances are $\sim 8 - 15\%$.

VI. CONCLUSIONS

In this paper, given first is a general formulation for deriving lower order moments of the one- and two-point correlation functions in eigenvalues that is valid for any embedded random matrix for fermions as well as for bosons with $U(\Omega) \otimes SU(r)$ embedding and with two-body interactions preserving $SU(r)$ symmetry. Results of the present paper unify all the results known before for EGUE(2)'s and BEGUE(2)'s. Presented are new results for boson systems with $SU(r)$ symmetry, $r = 2, 3$. These results should be useful in future studies of two species boson systems and spin one boson systems. In future, it will be useful to derive analytical forms for $SU(\Omega)$ Racah coefficients [6, 7] or develop tractable methods for their numerical evaluation to establish Gaussian form of the eigenvalue densities generated by embedded ensembles with $SU(r)$ symmetry both for boson and fermion systems.

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